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Finite-temperature Casimir pistons for an electromagnetic field with mixed boundary conditions and its classical limit

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Abstract

In this paper, the finite-temperature Casimir force acting on a two-dimensional Casimir piston due to an electromagnetic field is computed. It was found that if mixed boundary conditions are assumed on the piston and its opposite wall, then the Casimir force always tends to restore the piston toward the equilibrium position, regardless of the boundary conditions assumed on the walls transverse to the piston. In contrast, if pure boundary conditions are assumed on the piston and the opposite wall, then the Casimir force always tends to pull the piston toward the closer wall and away from the equilibrium position. The nature of the force is not affected by temperature. However, in the high-temperature regime, the magnitude of the Casimir force grows linearly with respect to temperature. This shows that the Casimir effect has a classical limit as has been observed in other literature.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Since the work of Cavalcanti [1], the Casimir effect of the piston geometry (see figure 1) has attracted considerable interest as it has been shown to be free of a divergence problem. Some studies have been devoted to this subject [2–16]. It was found that for a massless scalar field with periodic boundary conditions (b.c.), Dirichlet b.c. and Neumann b.c., and for an electromagnetic field with perfect electric conductor (PEC) b.c. and perfect magnetic conductor (PMC) b.c. in a d -dimensional space, the Casimir force acting on the piston always tends to pull the piston to the closest wall. This might create an undesirable effect known as stiction in the functionality of nano-devices. In [5], Barton showed that for a thin piston

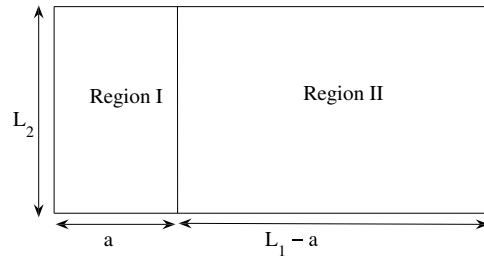


Figure 1. The two-dimensional rectangular piston.

with weakly reflecting dielectrics, the Casimir force at small separations is attractive, but turns to repulsive as the separation increases. Another scenario which leads to repulsive Casimir force was considered in [9, 16], where a massless or massive scalar field is assumed to satisfy Neumann b.c. on the piston and Dirichlet b.c. on all other walls. In this case, the zero-temperature Casimir force was shown to be always repulsive. In [10], it was suggested that a perfectly conducting piston inside a rectangular cavity with infinitely permeable walls will lead to repulsive Casimir force.

In this paper, we consider the thermal correction to the repulsive Casimir force due to an electromagnetic field with mixed boundary conditions (PEC b.c. on one wall and PMC b.c. on the opposite wall) and determine whether temperature will change the nature of the force. We only consider the case where the space dimension $d = 2$. This will simplify the mathematical computation but it gives enough indications for the general case of higher dimensions which will be considered in future. The two-dimensional rectangular Casimir pistons for the electromagnetic field with purely PEC b.c. and purely PMC b.c. have been studied. The Casimir effect due to an electromagnetic field with PMC b.c. coincides with the Casimir effect due to a massless scalar field with Dirichlet b.c. whose zero-temperature limit is studied in the pioneering work [1]. The Casimir effect due to an electromagnetic field with PEC b.c. coincides with the Casimir effect due to a massless scalar field with Neumann b.c. whose zero-temperature limit is considered in [8]. The finite-temperature Casimir effect was recently considered in [15]. It was found that for pure boundary conditions, the Casimir force is always attractive at any temperature. Therefore, it will be interesting to see whether the thermal correction affects the repulsive nature of the Casimir force due to the electromagnetic field with mixed b.c. This is the issue addressed in this paper. We consider a more general case of mixed b.c., where each pair of parallel plates can either assume pure boundary conditions (both PEC b.c. or both PMC b.c.) or mixed boundary conditions.

In this paper, we work in the units where \hbar (reduced Planck constant), c (speed of light) and k_B (Boltzmann constant) are equal to unity.

2. Casimir energy for an electromagnetic field with mixed boundary conditions inside a rectangular cavity

Recall that the finite-temperature Casimir energy is defined as the sum of the zero-temperature Casimir energy and the temperature correction, i.e.,

$$E_{\text{Cas}} = E_{\text{Cas}}^0 + \Delta E_{\text{Cas}} = \frac{1}{2} \sum_{\omega_k \neq 0} \omega_k + T \sum_{\omega_k \neq 0} \log(1 - e^{-\frac{\omega_k}{T}}),$$

where ω_k runs through all zero point energies. The sum corresponding to the temperature correction ΔE_{Cas} is a convergent sum. However, the zero-temperature contribution E_{Cas}^0 is divergent. There are different ways to regularize this sum. In the zeta regularization scheme [17–20], we define the zeta function

$$\zeta_0(s) = \sum_{\omega_k \neq 0} \omega_k^{-2s}$$

and analytically continue it to a neighborhood of $s = -1/2$. If $\zeta_0(s)$ is regular at $s = -1/2$, the zeta-regularized zero-temperature Casimir energy is then defined as

$$E_{\text{Cas}}^{0,\text{zetareg}} = \frac{1}{2} \zeta_0\left(-\frac{1}{2}\right).$$

Correspondingly, the finite-temperature Casimir energy can be computed by using the zeta function

$$\zeta(s) = \sum_{\omega_k \neq 0} \sum_{l=-\infty}^{\infty} (\omega_k^2 + (2\pi l T)^2)^{-s}.$$

It can be shown that (see [21–24]) if $\zeta(s)$ has an analytic continuation to a neighborhood of $s = 0$ with $\zeta(0) = 0$, then

$$\zeta'(0) = -\frac{1}{T} \zeta_0\left(-\frac{1}{2}\right) - 2 \sum_{\omega_k \neq 0} \log\left(1 - e^{-\frac{\omega_k}{T}}\right).$$

Consequently, the zeta-regularized finite-temperature Casimir energy is equal to

$$E_{\text{Cas}}^{\text{reg}} = -\frac{T}{2} \zeta'(0).$$

A disadvantage of applying the zeta regularization scheme is that all the divergence terms in the Casimir energy have been renormalized to zero. However, it can be shown as in [15] that in the piston scenario, the divergence terms of the Casimir force acting on the piston due to region I and region II always cancel without renormalization due to the fact that the divergence terms of the Casimir energies are linear in L_1 .

For an electromagnetic field inside a d -dimensional space Ω , the field strength is represented by a totally anti-symmetric rank-two tensor $\mathfrak{F}^{\mu\nu}$, $\mu, \nu = 0, 1, \dots, d$, satisfying the equations

$$\partial_\mu \tilde{\mathfrak{F}}^{\mu\nu_1 \dots \nu_{d-2}} = 0, \quad \partial_\mu \mathfrak{F}^{\mu\nu} = j^\nu, \tag{1}$$

where $\tilde{\mathfrak{F}}^{\mu_1 \dots \mu_{d-1}} = \varepsilon^{\mu_1 \dots \mu_{d-1}, \nu, \lambda} \mathfrak{F}_{\nu\lambda}$ is the dual tensor of $\mathfrak{F}^{\mu\nu}$, and j^ν is the current. In the vacuum state $j^\nu = 0$. There are two ideal boundary conditions that are of particular interest, i.e., the PEC b.c. characterized by $n_\mu \tilde{\mathfrak{F}}^{\mu\nu_1 \dots \nu_{d-2}}|_{\partial\Omega} = 0$ and the PMC b.c. characterized by $n_\mu \mathfrak{F}^{\mu\nu}|_{\partial\Omega} = 0$. Introducing the potentials A^μ so that

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu, \quad \partial^0 = \partial_0, \quad \partial^i = -\partial_i, \quad 1 \leq i \leq d,$$

and working in the radiation gauge

$$A^0 = 0, \quad \partial_i A^i = 0,$$

equation (1) is equivalent to

$$\Delta A^i = 0, \quad \Delta := \partial_0^2 - \sum_{j=1}^d \partial_j^2,$$

when $j^\mu = 0$. When the space Ω is a rectangular cavity $\Omega = [0, L_1] \times \dots \times [0, L_d]$, the PEC b.c. on a wall $x_i = 0$ or $x_i = L_i$ is equivalent to

$$\partial_\mu A_\nu - \partial_\nu A_\mu|_{x_i=0 \text{ or } x_i=L_i} = 0$$

for all $\mu \neq \nu \in \{0, 1, \dots, d\} \setminus \{i\}$; whereas the PMC b.c. is equivalent to

$$\partial_i A_\mu - \partial_\mu A_i|_{x_i=0 \text{ or } x_i=L_i} = 0$$

for all $\mu \in \{0, 1, \dots, d\} \setminus \{i\}$. Restricted to the case $d = 2$, we consider the following different combinations of boundary conditions:

Case I. Mixed boundary conditions (i.e., one wall PEC b.c. and one wall PMC b.c.) on both x_1 and x_2 directions.

Case II. Mixed boundary conditions on one direction, say x_1 , and purely PEC b.c. in the other direction.

Case III. Mixed boundary conditions on one direction, say x_1 , and purely PMC b.c. in the other direction.

Now we derive the finite-temperature Casimir energy of the electromagnetic field for each of the above boundary conditions:

Case I. In this case, we are looking for solutions of $A_1(x_1, x_2, t)$ and $A_2(x_1, x_2, t)$ satisfying

$$(\partial_t^2 - \partial_{x_1}^2 - \partial_{x_2}^2)A_i = 0, \quad i = 1, 2, \quad \partial_{x_1}A_1 + \partial_{x_2}A_2 = 0, \quad (2)$$

and the boundary conditions

$$\partial_t A_1|_{x_1=L_1, x_2=0} = 0, \quad \partial_t A_2|_{x_1=0, x_2=L_2} = 0, \quad (\partial_{x_1}A_2 - \partial_{x_2}A_1)|_{x_1=L_1, x_2=L_2} = 0.$$

It is easy to verify that a basis of solutions is given by

$$\begin{pmatrix} A_1(x_1, x_2, t) \\ A_2(x_1, x_2, t) \end{pmatrix} = \begin{pmatrix} \alpha_1 \cos \frac{\pi(k_1 + \frac{1}{2})x_1}{L_1} \sin \frac{\pi(k_2 + \frac{1}{2})x_2}{L_2} \\ \alpha_2 \sin \frac{\pi(k_1 + \frac{1}{2})x_1}{L_1} \cos \frac{\pi(k_2 + \frac{1}{2})x_2}{L_2} \end{pmatrix} e^{-\omega_k t}, \quad k_1, k_2 \in \tilde{\mathbb{N}} = \mathbb{N} \cup \{0\},$$

subjected to the condition

$$\frac{\alpha_1(k_1 + \frac{1}{2})}{L_1} + \frac{\alpha_2(k_2 + \frac{1}{2})}{L_2} = 0.$$

Here

$$\omega_k = \pi \sqrt{\left(\frac{k_1 + \frac{1}{2}}{L_1}\right)^2 + \left(\frac{k_2 + \frac{1}{2}}{L_2}\right)^2}.$$

The corresponding zeta function is

$$\zeta(s) = \frac{\pi^{-2s}}{4} \left\{ Z_3\left(s; \frac{1}{2L_1}, \frac{1}{2L_2}, 2T\right) - Z_3\left(s; \frac{1}{2L_1}, \frac{1}{L_2}, 2T\right) - Z_3\left(s; \frac{1}{L_1}, \frac{1}{2L_2}, 2T\right) + Z_3\left(s; \frac{1}{L_1}, \frac{1}{L_2}, 2T\right) \right\},$$

where $Z_n(s; c_1, \dots, c_n)$ is the homogeneous Epstein zeta function defined by

$$Z_n(s; c_1, \dots, c_n) = \sum_{\mathbf{k} \in \widehat{\mathbb{Z}}^n} \left(\sum_{j=1}^n [c_j k_j]^2 \right)^{-s}, \quad (3)$$

and $\widehat{\mathbb{Z}}^n = \mathbb{Z}^n \setminus \{\mathbf{0}\}$. Since $Z_n(0; c_1, \dots, c_n) = -1$, we find that the regularized Casimir energy for the electromagnetic field with mixed boundary conditions in both x_1 and x_2 directions of

a rectangular cavity is given by

$$E_{\text{Cas}}^{\text{I,reg}}(L_1, L_2) = -\frac{T}{8} \left\{ Z'_3 \left(0; \frac{1}{2L_1}, \frac{1}{2L_2}, 2T \right) - Z'_3 \left(0; \frac{1}{2L_1}, \frac{1}{L_2}, 2T \right) - Z'_3 \left(0; \frac{1}{L_1}, \frac{1}{2L_2}, 2T \right) + Z'_3 \left(0; \frac{1}{L_1}, \frac{1}{L_2}, 2T \right) \right\}. \quad (4)$$

Explicit formulae for $Z'_n(0; c_1, \dots, c_n)$ are given in the appendix.

Case II. in this case, we are looking for solutions of $A_1(x_1, x_2, t)$ and $A_2(x_1, x_2, t)$ satisfying (2) and the boundary conditions

$$\partial_t A_1|_{x_1=L_1, x_2=0, x_2=L_2} = 0, \quad \partial_t A_2|_{x_1=0} = 0, \quad (\partial_{x_1} A_2 - \partial_{x_2} A_1)|_{x_1=L_1} = 0.$$

A basis of solutions is given by

$$\begin{pmatrix} A_1(x_1, x_2, t) \\ A_2(x_1, x_2, t) \end{pmatrix} = \begin{pmatrix} \alpha_1 \cos \frac{\pi(k_1 + \frac{1}{2})x_1}{L_1} \sin \frac{\pi k_2 x_2}{L_2} \\ \alpha_2 \sin \frac{\pi(k_1 + \frac{1}{2})x_1}{L_1} \cos \frac{\pi k_2 x_2}{L_2} \end{pmatrix} e^{-\omega_k t}, \quad k_1, k_2 \in \tilde{\mathbb{N}},$$

$$\omega_k = \pi \sqrt{\left(\frac{k_1 + \frac{1}{2}}{L_1}\right)^2 + \left(\frac{k_2}{L_2}\right)^2},$$

subjected to the condition

$$\frac{\alpha_1(k_1 + \frac{1}{2})}{L_1} + \frac{\alpha_2 k_2}{L_2} = 0.$$

The corresponding regularized Casimir energy is

$$E_{\text{Cas}}^{\text{II,reg}}(L_1, L_2) = -\frac{T}{8} \left\{ Z'_3 \left(0; \frac{1}{2L_1}, \frac{1}{L_2}, 2T \right) - Z'_3 \left(0; \frac{1}{L_1}, \frac{1}{L_2}, 2T \right) + Z'_2 \left(0; \frac{1}{2L_1}, 2T \right) - Z'_2 \left(0; \frac{1}{L_1}, 2T \right) \right\}. \quad (5)$$

Case III. in this case, we are looking for solutions of $A_1(x_1, x_2, t)$ and $A_2(x_1, x_2, t)$ satisfying (2) and the boundary conditions

$$\partial_t A_1|_{x_1=L_1} = 0, \quad \partial_t A_2|_{x_1=0, x_2=0, x_2=L_2} = 0, \quad (\partial_{x_1} A_2 - \partial_{x_2} A_1)|_{x_1=L_1, x_2=0, x_2=L_2} = 0.$$

A basis of solutions is given by

$$\begin{pmatrix} A_1(x_1, x_2, t) \\ A_2(x_1, x_2, t) \end{pmatrix} = \begin{pmatrix} \alpha_1 \cos \frac{\pi(k_1 + \frac{1}{2})x_1}{L_1} \cos \frac{\pi k_2 x_2}{L_2} \\ \alpha_2 \sin \frac{\pi(k_1 + \frac{1}{2})x_1}{L_1} \sin \frac{\pi k_2 x_2}{L_2} \end{pmatrix} e^{-\omega_k t}, \quad k_1 \in \tilde{\mathbb{N}}, \quad k_2 \in \mathbb{N},$$

$$\omega_k = \pi \sqrt{\left(\frac{k_1 + \frac{1}{2}}{L_1}\right)^2 + \left(\frac{k_2}{L_2}\right)^2},$$

subjected to the condition

$$-\frac{\alpha_1(k_1 + \frac{1}{2})}{L_1} + \frac{\alpha_2 k_2}{L_2} = 0.$$

The corresponding regularized Casimir energy is

$$E_{\text{Cas}}^{\text{III,reg}}(L_1, L_2) = -\frac{T}{8} \left\{ Z_3' \left(0; \frac{1}{2L_1}, \frac{1}{L_2}, 2T \right) - Z_3' \left(0; \frac{1}{L_1}, \frac{1}{L_2}, 2T \right) - Z_2' \left(0; \frac{1}{2L_1}, 2T \right) + Z_2' \left(0; \frac{1}{L_1}, 2T \right) \right\}. \quad (6)$$

Note that there is a slight difference between the set of eigenmodes in the case III. In the case II, we allow $k_2 = 0$ which corresponds to solutions

$$\begin{pmatrix} A_1(x_1, x_2, t) \\ A_2(x_1, x_2, t) \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha_2 \sin \frac{\pi(k_1 + \frac{1}{2})x_1}{L_1} \end{pmatrix} e^{-\omega_k t}, \quad k_1 \in \tilde{\mathbb{N}}, \quad \omega_k = \frac{\pi(k_1 + \frac{1}{2})}{L_1}.$$

However, in the case III, $k_2 = 0$ implies that $\alpha_1 = 0$ and $A_1 = A_2 \equiv 0$. Therefore there is no eigenmode with $k_2 = 0$.

3. Casimir force acting on the piston for an electromagnetic field with mixed boundary conditions

In this section, we consider the Casimir force acting on a two-dimensional rectangular piston due to an electromagnetic field with mixed boundary conditions. The boundary conditions on the walls of region I are the cases I, II, III as considered in the previous section. In region II, we assume that the boundary condition on the wall $x_1 = L_1$ is the same as on the wall $x_1 = 0$. We have the following cases.

3.1. Case MBC-A

We assume mixed boundary conditions on both directions. In this case, we find that the total regularized Casimir energy of the piston system is

$$E_{\text{Cas}}^{\text{A,reg}}(a; L_1, L_2) = E_{\text{Cas}}^{\text{I,reg}}(a, L_2) + E_{\text{Cas}}^{\text{I,reg}}(L_1 - a, L_2).$$

Applying the Chowla–Selberg formula (A.2) to (4), we find that

$$E_{\text{Cas}}^{\text{I,reg}}(L_1, L_2) = -\frac{T}{8} \left\{ \frac{L_1 L_2}{2\pi T} Z_2 \left(\frac{3}{2}; 2L_2, \frac{1}{2T} \right) - \frac{L_1 L_2}{4\pi T} Z_2 \left(\frac{3}{2}; L_2, \frac{1}{2T} \right) + 4 \sum_{k_1=1}^{\infty} \sum_{k_2=0}^{\infty} \sum_{l=-\infty}^{\infty} \frac{(-1)^{k_1}}{k_1} \exp \left(-2\pi k_1 L_1 \sqrt{\left(\frac{k_2 + \frac{1}{2}}{L_2} \right)^2 + (2lT)^2} \right) \right\}.$$

Therefore, in the limit $L_1 \rightarrow \infty$, the Casimir force acting on the piston is given by

$$\begin{aligned} F_{\text{Cas}}^{\text{A}, L_1=\infty}(a; L_2) &= \lim_{L_1 \rightarrow \infty} F_{\text{Cas}}^{\text{A}}(a; L_1, L_2) \\ &= - \lim_{L_1 \rightarrow \infty} \frac{\partial}{\partial a} E_{\text{Cas}}^{\text{A,reg}}(a; L_1, L_2) \\ &= \pi T \sum_{k_2=0}^{\infty} \sum_{l=-\infty}^{\infty} \frac{\sqrt{\left(\frac{k_2 + \frac{1}{2}}{L_2} \right)^2 + (2lT)^2}}{\exp \left(2\pi a \sqrt{\left(\frac{k_2 + \frac{1}{2}}{L_2} \right)^2 + (2lT)^2} \right) + 1}. \end{aligned} \quad (7)$$

Note that this is a positive decreasing function in a . Consequently, when L_1 is finite, the Casimir force acting on the piston,

$$F_{\text{Cas}}^A(a; L_1, L_2) = F_{\text{Cas}}^{A, L_1=\infty}(a; L_2) - F_{\text{Cas}}^{A, L_1=\infty}(L_1 - a; L_2),$$

is positive if $a < L_1 - a$, and is negative if $a > L_1 - a$. In other words, at any temperature, the Casimir force always tends to restore the piston to the equilibrium position $x_1 = L_1/2$, which is the middle of the cavity.

The infinite summation in expression (7) for the Casimir force converges very fast if $a \gg L_2$. It shows that in the limit $L_1 \rightarrow \infty$, the magnitude of the Casimir force decays exponentially when the plate separation a is large. In the most practical situation, we are interested in the opposite case where $a \ll L_2$. In this latter case, the Chowla–Selberg formula (A.2) gives

$$F_{\text{Cas}}^{A, L_1=\infty}(a; L_2) = \frac{3\zeta_R(3)}{32\pi} \frac{L_2}{a^3} - \frac{L_2}{32\pi} \sum_{(k_3, \ell) \in \widehat{\mathbb{Z}}^2} \frac{(-1)^{k_2}}{([k_2 L_2]^2 + [\frac{l}{2T}]^2)^{\frac{3}{2}}} + \frac{\pi L_2}{2a^3} \sum_{k_1=0}^{\infty} \sum_{(k_2, \ell) \in \widehat{\mathbb{Z}}^2} (-1)^{k_2} \times \left(k_1 + \frac{1}{2}\right)^2 K_0 \left(\frac{2\pi(k_1 + \frac{1}{2})}{a} \sqrt{[k_2 L_2]^2 + \left[\frac{l}{2T}\right]^2}\right). \tag{8}$$

This shows that at any temperature, when the plate separation a is small, the leading behavior of the Casimir force is given by

$$F_{\text{Cas}}^{A, L_1=\infty}(a; L_2) \sim \frac{3\zeta_R(3)}{32\pi} \frac{L_2}{a^3} + O(a^0).$$

It implies that when $a \rightarrow 0^+$, the magnitude of the Casimir force approaches ∞ and behaves as $1/a^3$. From this we can conclude that at any temperature, the Casimir force acting on the piston, considered as a function of $a \in (0, L_1)$, decreases from ∞ to 0 when $a \in (0, L_1/2)$ and increases from 0 to ∞ when $a \in (L_1/2, L_1)$.

Formula (7) can also be used to study the high-temperature behavior of the Casimir force. It shows that in the high-temperature regime, the leading behavior of the Casimir force is

$$F_{\text{Cas}}^A(a; L_1, L_2) \sim \frac{\pi T}{L_2} \sum_{k_2=0}^{\infty} \frac{k_2 + \frac{1}{2}}{\exp\left(\frac{2\pi a}{L_2} \left(k_2 + \frac{1}{2}\right)\right) + 1} - (a \longleftrightarrow L_1 - a), \tag{9}$$

which is linear in T . The remaining terms decay exponentially as $T \rightarrow \infty$. If we restore the units \hbar, c and k_B to the expression for Casimir force, we find that a term with T^j will be accompanied by \hbar^{j-1} . Therefore, (9) shows that the Casimir force acting on the piston has a classical ($\hbar \rightarrow 0$) limit, as has also been observed in other works on the Casimir effect (see, e.g., [25–28]). The right-hand side of (9) is called the classical term of the Casimir force.

In the low-temperature ($T \ll 1$) regime, the Casimir force is dominated by the zero-temperature Casimir force, with the correction term being the temperature correction:

$$F_{\text{Cas}}^A(a; L_1, L_2) = F_{\text{Cas}}^{A, T=0}(a; L_1, L_2) + \Delta_T F_{\text{Cas}}^A(a; L_1, L_2).$$

Applying the Chowla–Selberg formula (A.1), we have

$$\begin{aligned} -\frac{L_2}{32\pi} \sum_{(k_3, \ell) \in \widehat{\mathbb{Z}}^2} \frac{(-1)^{k_2}}{([k_2 L_2]^2 + [\frac{l}{2T}]^2)^{\frac{3}{2}}} &= -\frac{L_2}{32\pi} \left(2Z_2\left(\frac{3}{2}; 2L_2, \frac{1}{2T}\right) - Z_2\left(\frac{3}{2}; L_2, \frac{1}{2T}\right)\right) \\ &= \frac{3\zeta_R(3)}{64\pi L_2^2} - \frac{T}{L_2} \sum_{k_2=0}^{\infty} \sum_{l=1}^{\infty} \frac{k_2 + \frac{1}{2}}{l} K_1\left(\frac{\pi(k_2 + \frac{1}{2})l}{L_2 T}\right). \end{aligned}$$

With this, we can read from the formula (8) that the zero-temperature Casimir force is given by

$$F_{\text{Cas}}^{A,T=0}(a; L_1, L_2) = \frac{3\zeta_R(3)}{32\pi} \frac{L_2}{a^3} + \frac{3\zeta_R(3)}{64\pi L_2^2} + \frac{\pi L_2}{a^3} \sum_{k_1=0}^{\infty} \sum_{k_2=1}^{\infty} (-1)^{k_2} \left(k_1 + \frac{1}{2}\right)^2 \times K_0\left(\frac{2\pi k_2 (k_1 + \frac{1}{2}) L_2}{a}\right) - (a \longleftrightarrow L_1 - a),$$

and the thermal correction is

$$\Delta_T F_{\text{Cas}}^A(a; L_1, L_2) = -\frac{T}{L_2} \sum_{k_2=0}^{\infty} \sum_{l=1}^{\infty} \frac{k_2 + \frac{1}{2}}{l} K_1\left(\frac{\pi (k_2 + \frac{1}{2}) l}{L_2 T}\right) + \frac{\pi L_2}{a^3} \sum_{k_1=0}^{\infty} \sum_{k_2=-\infty}^{\infty} \sum_{l=1}^{\infty} \times (-1)^{k_2} \left(k_1 + \frac{1}{2}\right)^2 K_0\left(\frac{2\pi (k_1 + \frac{1}{2})}{a} \sqrt{[k_2 L_2]^2 + \left[\frac{l}{2T}\right]^2}\right) - (a \leftrightarrow L_1 - a).$$

Note that if $L_1 \rightarrow \infty$, the thermal correction to the Casimir force decays to zero exponentially fast when $T \rightarrow 0^+$.

In the limit $L_1, L_2 \rightarrow \infty$, the geometric configuration becomes that of a pair of infinite parallel plates separated by a distance a . In this case, since

$$-\frac{L_2}{32\pi} \sum_{(k_3, \ell) \in \widehat{\mathbb{Z}}^2} \frac{(-1)^{k_2}}{([k_2 L_2]^2 + [\frac{l}{2T}]^2)^{\frac{3}{2}}} = -\frac{L_2 T^3}{2\pi} \zeta_R(3) + \frac{\pi}{48} \frac{T}{L_2} - 2T^2 \sum_{k_2=1}^{\infty} \sum_{l=1}^{\infty} \frac{(-1)^{k_2}}{k_2} l K_1(4\pi k_2 l L_2 T),$$

equation (8) then implies that in the infinite parallel plates limit, the Casimir force acting on a wall is given by

$$F_{\text{Cas}}^{A,\parallel}(a) = L_2 \left\{ \frac{3\zeta_R(3)}{32\pi a^3} - \frac{T^3}{2\pi} \zeta_R(3) + \frac{\pi}{a^3} \sum_{k_1=0}^{\infty} \sum_{l=1}^{\infty} \left(k_1 + \frac{1}{2}\right)^2 K_0\left(\frac{\pi l (k_1 + \frac{1}{2})}{a T}\right) \right\}. \quad (10)$$

This shows that for infinite parallel plates, the zero-temperature Casimir force is

$$F_{\text{Cas}}^{A,\parallel,T=0}(a) = \frac{3\zeta_R(3)}{32\pi a^3} L_2.$$

The temperature correction is of order T^3 as $T \rightarrow 0^+$. The remaining terms decay to zero exponentially fast when $T \rightarrow 0^+$. In the high temperature regime,

$$F_{\text{Cas}}^{A,\parallel}(a) = L_2 \left\{ \frac{\pi}{48a^2} T - \frac{2T^2}{a} \sum_{k_1=1}^{\infty} \sum_{l=1}^{\infty} (-1)^{k_1} \frac{l}{k_1} K_1(4\pi l k_1 T a) - 8\pi T^3 \sum_{k_1=1}^{\infty} \sum_{l=1}^{\infty} (-1)^{k_1} l^2 K_0(4\pi l k_1 T a) \right\}.$$

This shows that the classical limit of the Casimir force acting on a pair of infinite parallel plates with mixed boundary conditions is

$$F_{\text{Cas}}^{A,\parallel,\text{classical}}(a) = \frac{\pi L_2}{48a^2} T.$$

3.2. Case MBC-B

We assume mixed boundary conditions in the x_1 direction and purely PEC b.c. in the x_2 direction. Using the same method as the previous section, we find that the Casimir force acting on the piston is given by

$$F_{\text{Cas}}^{\text{B}}(a; L_1, L_2) = F_{\text{Cas}}^{\text{B}, L_1=\infty}(a; L_2) - F_{\text{Cas}}^{\text{B}, L_1=\infty}(L_1 - a; L_2),$$

where

$$F_{\text{Cas}}^{\text{B}, L_1=\infty}(a; L_2) = \pi T \sum_{(k_2, l) \in \mathbb{N} \times \mathbb{Z} \setminus \{0\}} \frac{\sqrt{\left(\frac{k_2}{L_2}\right)^2 + (2lT)^2}}{\exp\left(2\pi a \sqrt{\left(\frac{k_2}{L_2}\right)^2 + (2lT)^2}\right) + 1}. \quad (11)$$

As in the previous case, this shows that at any temperature, the Casimir force tends to pull the piston to the equilibrium position $x_1 = L_1/2$. Moreover, it shows that in the high-temperature limit, the leading term of the Casimir force is given by the classical term

$$F_{\text{Cas}}^{\text{B}}(a; L_1, L_2) \sim \frac{\pi T}{L_2} \sum_{k_2=1}^{\infty} \frac{k_2}{\exp\left(\frac{2\pi k_2 a}{L_2}\right) + 1} - (a \leftrightarrow L_1 - a). \quad (12)$$

An alternative expression for $F_{\text{Cas}}^{\text{B}, L_1=\infty}(a; L_2)$ that can be used to study the small a and low T behavior of the Casimir force is

$$\begin{aligned} F_{\text{Cas}}^{\text{B}, L_1=\infty}(a; L_2) &= \frac{3\zeta_R(3)}{32\pi} \frac{L_2}{a^3} + \frac{\pi}{96a^2} - \frac{\zeta_R(3)}{16\pi L_2^2} - \frac{\pi T^2}{6} - \frac{T}{L_2} \sum_{k_2=1}^{\infty} \sum_{l=1}^{\infty} \frac{k_2}{l} K_1\left(\frac{\pi k_2 l}{L_2 T}\right) \\ &+ \frac{\pi L_2}{2a^3} \sum_{k_1=0}^{\infty} \sum_{(k_2, \ell) \in \widehat{\mathbb{Z}}^2} \left(k_1 + \frac{1}{2}\right)^2 K_0\left(\frac{2\pi\left(k_1 + \frac{1}{2}\right)}{a} \sqrt{[k_2 L_2]^2 + \left[\frac{l}{2T}\right]^2}\right) \\ &+ \frac{\pi}{2a^2} \sum_{k_1=0}^{\infty} \frac{\left(k_1 + \frac{1}{2}\right)}{\exp\left(\frac{\pi\left(k_1 + \frac{1}{2}\right)}{Ta}\right) - 1}. \end{aligned} \quad (13)$$

It shows that when the plate separation a is small, the leading term of the Casimir force is given by

$$F_{\text{Cas}}^{\text{B}}(a; L_1, L_2) \sim \frac{3\zeta_R(3)}{32\pi} \frac{L_2}{a^3} + \frac{\pi}{96a^2} + O(a^0).$$

Note that the first term behaves as $1/a^3$ when $a \rightarrow 0^+$. On the other hand, (13) gives the zero-temperature Casimir force as

$$\begin{aligned} F_{\text{Cas}}^{\text{B}, T=0}(a; L_1, L_2) &= \frac{3\zeta_R(3)}{32\pi} \frac{L_2}{a^3} + \frac{\pi}{96a^2} - \frac{\zeta_R(3)}{16\pi L_2^2} + \frac{\pi L_2}{a^3} \sum_{k_1=0}^{\infty} \sum_{k_2=1}^{\infty} \left(k_1 + \frac{1}{2}\right)^2 \\ &\times K_0\left(\frac{2\pi\left(k_1 + \frac{1}{2}\right)k_2 L_2}{a}\right) - (a \leftrightarrow L_1 - a). \end{aligned} \quad (14)$$

The thermal correction goes to zero exponentially fast when $T \rightarrow 0^+$.

In the parallel plate limit, it can be checked that one would obtain the same result as (10). This should be expected since in the limit $L_2 \rightarrow \infty$, the boundary conditions assumed on the x_2 direction become immaterial.

3.3. Case MBC-C

We assume mixed boundary conditions in the x_1 direction and purely PMC b.c. in the x_2 direction. This case is very similar to the MBC-B case. We find that the Casimir force acting on the piston is given by

$$F_{\text{Cas}}^{\text{C}}(a; L_1, L_2) = F_{\text{Cas}}^{\text{C}, L_1=\infty}(a; L_2) - F_{\text{Cas}}^{\text{C}, L_1=\infty}(L_1 - a; L_2),$$

where

$$F_{\text{Cas}}^{\text{C}, L_1=\infty}(a; L_2) = \pi T \sum_{k_2=1}^{\infty} \sum_{l=-\infty}^{\infty} \frac{\sqrt{\left(\frac{k_2}{L_2}\right)^2 + (2lT)^2}}{\exp\left(2\pi a \sqrt{\left(\frac{k_2}{L_2}\right)^2 + (2lT)^2}\right) + 1}. \quad (15)$$

The difference between this term and the corresponding term in the case of MBC-B lies in the summation over k_2 , where now k_2 starts from 1 instead of 0. As in the previous case, equation (15) shows that at any temperature, the Casimir force tends to pull the piston to the equilibrium position $x_1 = L_1/2$. Moreover, it shows that in the high-temperature limit, the leading term of the Casimir force is given by the classical term

$$F_{\text{Cas}}^{\text{C}}(a; L_1, L_2) \sim \frac{\pi T}{L_2} \sum_{k_2=1}^{\infty} \frac{k_2}{\exp\left(\frac{2\pi k_2 a}{L_2}\right) + 1} - (a \leftrightarrow L_1 - a).$$

One can note that this classical term is the same as in the case of MBC-B given by (12). In other words, the difference between the Casimir forces for the cases MBC-B and MBC-C is insignificant at high temperature.

An alternative expression for $F_{\text{Cas}}^{\text{C}, L_1=\infty}(a; L_2)$ that can be used to study the small a and low T behavior of the Casimir force is

$$\begin{aligned} F_{\text{Cas}}^{\text{C}, L_1=\infty}(a; L_2) &= \frac{3\zeta_R(3)}{32\pi} \frac{L_2}{a^3} - \frac{\pi}{96a^2} - \frac{\zeta_R(3)}{16\pi L_2^2} - \frac{T}{L_2} \sum_{k_2=1}^{\infty} \sum_{l=1}^{\infty} \frac{k_2}{l} K_1\left(\frac{\pi k_2 l}{L_2 T}\right) \\ &+ \frac{\pi L_2}{2a^3} \sum_{k_1=0}^{\infty} \sum_{(k_2, \ell) \in \widehat{\mathbb{Z}}^2} \left(k_1 + \frac{1}{2}\right)^2 K_0\left(\frac{2\pi\left(k_1 + \frac{1}{2}\right)}{a} \sqrt{[k_2 L_2]^2 + \left[\frac{l}{2T}\right]^2}\right) \\ &- \frac{\pi}{2a^2} \sum_{k_1=0}^{\infty} \frac{\left(k_1 + \frac{1}{2}\right)}{\exp\left(\frac{\pi\left(k_1 + \frac{1}{2}\right)}{Ta}\right) - 1}. \end{aligned} \quad (16)$$

When the plate separation a is small, the leading terms of the Casimir force are given by

$$F_{\text{Cas}}^{\text{C}}(a; L_1, L_2) \sim \frac{3\zeta_R(3)}{32\pi} \frac{L_2}{a^3} - \frac{\pi}{96a^2} + O(a^0),$$

with leading order $1/a^3$ when $a \rightarrow 0^+$. On the other hand, the zero-temperature Casimir force is

$$\begin{aligned} F_{\text{Cas}}^{\text{C}, T=0}(a; L_1, L_2) &= \frac{3\zeta_R(3)}{32\pi} \frac{L_2}{a^3} - \frac{\pi}{96a^2} - \frac{\zeta_R(3)}{16\pi L_2^2} + \frac{\pi L_2}{a^3} \sum_{k_1=0}^{\infty} \sum_{k_2=1}^{\infty} \left(k_1 + \frac{1}{2}\right)^2 \\ &\times K_0\left(\frac{2\pi\left(k_1 + \frac{1}{2}\right)k_2 L_2}{a}\right) - (a \longleftrightarrow L_1 - a), \end{aligned} \quad (17)$$

which only differs with the MBC-B case by the sign of the term $\pi/(96a^2)$. The thermal correction also goes to zero exponentially fast when $T \rightarrow 0^+$.

We would like to remark that the regularized Casimir energy and Casimir force acting on the piston in this case are the same as the corresponding quantities for the massless scalar field which assume Neumann boundary condition on the piston and Dirichlet boundary conditions on the other walls. In fact, the zero-temperature Casimir force (17) agrees with the corresponding result in [9].

3.4. Case MBC-D

We assume PEC b.c. in the x_1 direction and mixed boundary conditions in the x_2 direction. In this case

$$E_{\text{Cas}}^{D,\text{reg}}(a; L_1, L_2) = E_{\text{Cas}}^{\text{II},\text{reg}}(L_2, a) + E_{\text{Cas}}^{\text{II},\text{reg}}(L_2, L_1 - a).$$

Similar computations give

$$F_{\text{Cas}}^D(a; L_1, L_2) = F_{\text{Cas}}^{D,L_1=\infty}(a; L_2) - F_{\text{Cas}}^{D,L_1=\infty}(L_1 - a; L_2),$$

where

$$F_{\text{Cas}}^{D,L_1=\infty}(a; L_2) = -\pi T \sum_{k_2=0}^{\infty} \sum_{l=-\infty}^{\infty} \frac{\sqrt{\left(\frac{k_2+\frac{1}{2}}{L_2}\right)^2 + (2lT)^2}}{\exp\left(2\pi a \sqrt{\left(\frac{k_2+\frac{1}{2}}{L_2}\right)^2 + (2lT)^2}\right) - 1}. \quad (18)$$

Contrary to the previous cases, now we find that the Casimir force acting on the piston always tends to pull the piston toward the closer wall, and away from the equilibrium position. Equation (18) also shows that in the high-temperature regime, the Casimir force is dominated by the classical term, i.e.

$$F_{\text{Cas}}^D(a; L_1, L_2) \sim -\frac{\pi T}{L_2} \sum_{k_2=0}^{\infty} \frac{k_2 + \frac{1}{2}}{\exp\left(\frac{2\pi a(k_2+\frac{1}{2})}{L_2}\right) - 1} - (a \leftrightarrow L_1 - a),$$

as $T \rightarrow \infty$. The remaining terms decay exponentially.

An alternative expression for the Casimir force is given by

$$\begin{aligned} F_{\text{Cas}}^D(a; L_1, L_2) = & -\frac{L_2}{8\pi a^3} \zeta_R(3) + \frac{3\zeta_R(3)}{64\pi L_2^2} - \frac{T}{L_2} \sum_{k_2=0}^{\infty} \sum_{l=1}^{\infty} \frac{k_2 + \frac{1}{2}}{l} K_1\left(\frac{\pi(k_2 + \frac{1}{2})l}{L_2 T}\right) + \frac{\pi L_2}{2a^3} \\ & \times \sum_{k_1=1}^{\infty} \sum_{(k_2, l) \in \widehat{\mathbb{Z}}^2} (-1)^{k_2} k_1^2 K_0\left(\frac{2\pi k_1}{a} \sqrt{(k_2 L_2)^2 + \left(\frac{l}{2T}\right)^2}\right) - (a \leftrightarrow L_1 - a). \end{aligned} \quad (19)$$

This shows that when the plate separation is small, the leading term of the Casimir force is

$$F_{\text{Cas}}^D(a; L_1, L_2) \sim -\frac{L_2}{8\pi a^3} \zeta_R(3) + O(a^0),$$

which is of order $1/a^3$. Equation (19) also shows that in the low-temperature limit, the Casimir force is dominated by the zero-temperature Casimir force given by

$$\begin{aligned} F_{\text{Cas}}^{D,T=0}(a; L_1, L_2) = & -\frac{L_2}{8\pi a^3} \zeta_R(3) + \frac{3\zeta_R(3)}{64\pi L_2^2} \\ & + \frac{\pi L_2}{a^3} \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} (-1)^{k_2} k_1^2 K_0\left(\frac{2\pi k_1 k_2 L_2}{a}\right) - (a \leftrightarrow L_1 - a). \end{aligned}$$

The thermal correction terms tend to zero exponentially fast when $T \rightarrow 0^+$.

In the infinite parallel plate limit, we find that

$$F_{\text{Cas}}^{\text{D},\parallel}(a) = L_2 \left\{ -\frac{\zeta_R(3)}{8\pi a^3} - \frac{T^3}{2\pi} \zeta_R(3) + \frac{\pi}{a^3} \sum_{k_1=1}^{\infty} \sum_{l=1}^{\infty} k_1^2 K_0\left(\frac{\pi l k_1}{aT}\right) \right\}, \quad (20)$$

which gives the zero-temperature Casimir force as

$$F_{\text{Cas}}^{\text{D},\parallel}(a) = -\frac{\zeta_R(3)L_2}{8\pi a^3},$$

agreeing with well-known results (see, e.g., [29]). An alternative expression for (20) is given by

$$F_{\text{Cas}}^{\text{D},\parallel}(a) = L_2 \left\{ -\frac{\pi T}{24a^2} - \frac{2T^2}{a} \sum_{k_1=1}^{\infty} \sum_{l=1}^{\infty} \frac{l}{k_1} K_1(4\pi l k_1 T a) - 8\pi T^3 \sum_{k_1=1}^{\infty} \sum_{l=1}^{\infty} l^2 K_0(4\pi l k_1 T a) \right\},$$

which shows that the classical limit of the Casimir force is given by

$$F_{\text{Cas}}^{\text{D},\parallel,\text{classical}}(a) = -\frac{\pi L_2}{24a^2} T.$$

3.5. Case MBC-E

We assume PMC b.c. on the x_1 direction and mixed boundary conditions on the x_2 direction. In this case, although the regularized Casimir energy is different from the regularized Casimir energy for the case MBC-D, one can verify that their difference is a term independent of L_1 . Consequently, the Casimir force acting on the piston for the case MBC-E is identical to that for the case MBC-D.

We do not discuss the cases where the electromagnetic field assumes purely PEC b.c. on both directions or assumes purely PMC b.c. on both directions. This has been considered in [15]. Another case we do not consider here is the case where the field assumes purely PEC b.c. on one direction and purely PMC b.c. on the other direction. The result is not much different from the cases of purely PEC b.c. or purely PMC b.c. on all directions.

4. Discussion and conclusion

We have computed the exact formulae for the finite-temperature Casimir force acting on a two-dimensional rectangular piston due to an electromagnetic field with different combinations of boundary conditions. From the results, we can conclude that if mixed boundary conditions are assumed on the piston and its opposite wall, then the Casimir force always tends to move the piston to the equilibrium position, regardless of the boundary conditions assumed on the perpendicular walls. In contrast, if purely PMC b.c. or purely PEC b.c. is assumed on the piston and its opposite wall, then the Casimir force always tends to move the piston toward the closer wall, again regardless of the boundary conditions assumed on the perpendicular walls. This nature of the force is not affected by the change of temperature. However, as in the case of pure boundary conditions discussed in [15], the magnitude of the Casimir force grows linearly with temperature when the temperature is high enough. This implies that although the Casimir force is a quantum effect, it has a classical limit, as has been observed in [25–28].

Comparing the magnitude of the Casimir force for various boundary conditions, we note that in the case of an open piston ($L_1 \rightarrow \infty$), the magnitude of the Casimir force always decreases as the plate separation a increases. Moreover, we see that when the plate separation

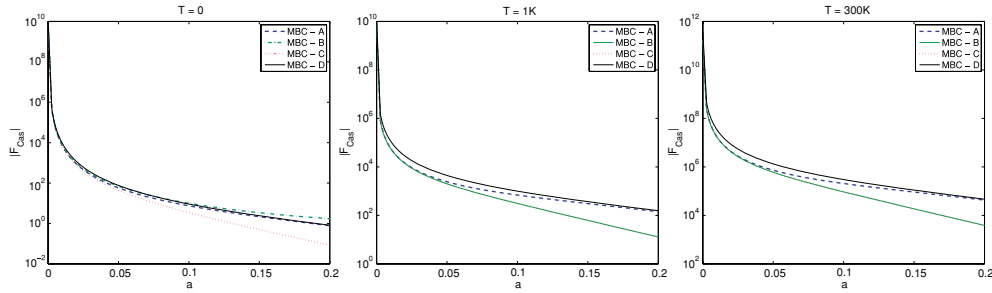


Figure 2. The absolute values of the Casimir forces for various boundary conditions when $L_1 \rightarrow \infty$ and $L_2 = 0.2$ m. Here the unit of a is m, the unit of force is $\hbar c \times \text{N}$. The graphs correspond to $T = 0$ K, 1 K and $T = 300$ K, respectively.

a is small, the leading term of the Casimir force for the cases MBC-A, MBC-B and MBC-C which all assume mixed boundary conditions in the x_1 direction are the same is equal to

$$\frac{3\zeta_R(3)}{32\pi} \frac{L_2}{a^3}. \tag{21}$$

For the cases MBC-D or MBC-E which assume pure boundary conditions in the x_1 direction, the leading term is

$$-\frac{\zeta_R(3)}{8\pi} \frac{L_2}{a^3}. \tag{22}$$

Its magnitude is 4/3 times larger than the case of mixed boundary conditions in the x_1 direction. Equations (21) and (22) are also the corresponding zero-temperature Casimir force in the infinite parallel plate limit. On the other hand, we also note that the classical limit of the Casimir force for the MBC-B and MBC-C cases is the same. For all the boundary conditions considered, the magnitude of the classical limit always decreases as the piston moves toward the equilibrium position. In the infinite parallel plate limit, the magnitude of the classical term for plates with mixed boundary conditions is half that for plates with pure boundary conditions. The comparisons of the Casimir forces with different boundary conditions and at different temperatures are depicted in figures 2 and 3, respectively. Here we would like to remark that by restoring the units \hbar , k_B and c , we have to replace T in the expressions for Casimir force with $k_B T / (\hbar c)$. Therefore, for physical $T = 1$ K we need to substitute $T = 436.7 \text{ m}^{-1}$ which is actually large if compared to a in the range $0.01 \text{ m} \sim 0.2 \text{ m}$ which is equivalent to a^{-1} in the range $5 \text{ m}^{-1} \sim 100 \text{ m}^{-1}$. This explains the big difference between the zero-temperature Casimir force and the Casimir force at $T = 1$ K observed in figure 3 when a is in the range $0.01 \text{ m} \sim 0.2 \text{ m}$. In fact, if we plot the Casimir force for a in the range < 0.1 mm, we would not observe significant difference between the Casimir force at $T = 0$ K and $T = 1$ K.

This work can be generalized to higher dimensions, where the formulae are expected to be more complicated. Moreover, there will be more different combinations of boundary conditions. We leave this discussions to the future. Another interesting subject to explore is to consider a ‘continuous’ change of boundary conditions from PEC b.c. to PMC b.c. on the piston but fixing the boundary condition on the opposite wall, and to investigate the gradual change of the nature of the Casimir force on the piston. This may give us some insights into the mechanism of the change of the nature of the Casimir force.

Finally, we would like to remark that although the piston scenario has the advantage of providing a formalism to obtain a Casimir force that is free of a divergence problem, it has its

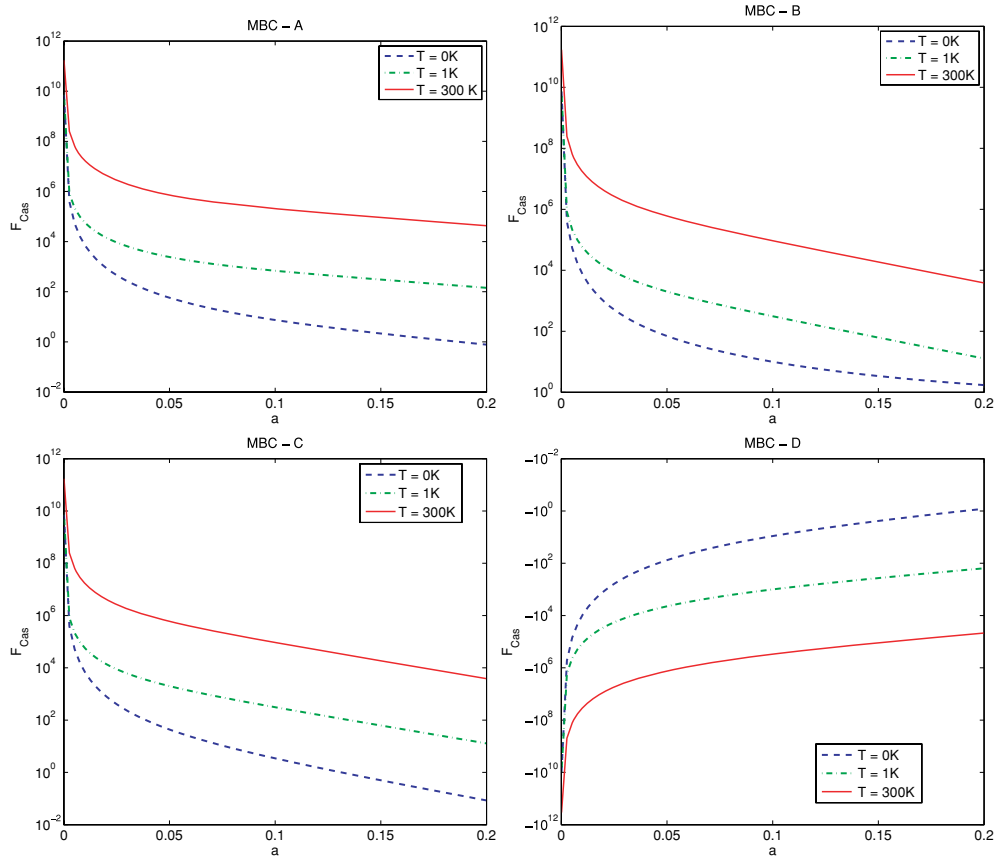


Figure 3. The comparisons of the Casimir forces at different temperatures when $L_1 \rightarrow \infty$ and $L_2 = 0.2$ m. Here the unit of a is m, the unit of force is $\hbar c \times \text{N}$.

own limitations. At the moment, this formalism cannot be used to obtain the Casimir force acting on the rectangular walls without substantial modification, otherwise it will lead to a thermodynamically inconsistent Casimir effect. Some recent endeavors to solve the problem of obtaining physically consistent Casimir force acting on the walls of a rectangular cavity can be found in [30, 31]. In particular, Geyer *et al* [30] have proposed a formalism that can give a thermodynamically consistent Casimir energy in an ideal rectangular metallic box.

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Appendix. The Chowla–Selberg formula for the Epstein zeta function and its derivative at zero

Here we gather some formulae we need for the Epstein zeta function (3) and its derivative at zero. The Chowla–Selberg formula [18–20, 32–37] says that

$$\begin{aligned}
 Z_n(s; c_1, \dots, c_n) &= Z_m(s; c_1, \dots, c_m) + \frac{\pi^{m/2} \Gamma\left(s - \frac{m}{2}\right)}{\left[\prod_{j=1}^m c_j\right] \Gamma(s)} Z_{n-m}\left(s - \frac{m}{2}; c_{m+1}, \dots, c_n\right) \\
 &+ \frac{1}{\Gamma(s)} \frac{2\pi^s}{\left[\prod_{j=1}^m c_j\right]} \sum_{\mathbf{k} \in \widehat{\mathbb{Z}^m \times \mathbb{Z}^{n-m}}} \left(\frac{\sum_{j=1}^m \left[\frac{k_j}{c_j}\right]^2}{\sum_{j=m+1}^n [c_j k_j]^2} \right)^{\frac{2s-m}{4}} \\
 &\times K_{s-\frac{m}{2}} \left(2\pi \sqrt{\left(\sum_{j=1}^m \left[\frac{k_j}{c_j}\right]^2 \right) \left(\sum_{j=m+1}^n [c_j k_j]^2 \right)} \right), \tag{A.1}
 \end{aligned}$$

where $K_\nu(z)$ is the modified Bessel function. By taking derivative with respect to s and setting $s = 0$, we find that

$$\begin{aligned}
 Z'_n(0; c_1, \dots, c_n) &= Z'_m(0; c_1, \dots, c_m) + \frac{\pi^{-n/2} \Gamma\left(\frac{n}{2}\right)}{\left[\prod_{j=1}^n c_j\right]} Z_{n-m}\left(\frac{n}{2}; \frac{1}{c_{m+1}}, \dots, \frac{1}{c_n}\right) \\
 &+ \frac{2}{\left[\prod_{j=1}^m c_j\right]} \sum_{\mathbf{k} \in \widehat{\mathbb{Z}^m \times \mathbb{Z}^{n-m}}} \left(\frac{\sum_{j=1}^m \left[\frac{k_j}{c_j}\right]^2}{\sum_{j=m+1}^n [c_j k_j]^2} \right)^{-\frac{m}{4}} \\
 &\times K_{\frac{m}{2}} \left(2\pi \sqrt{\left(\sum_{j=1}^m \left[\frac{k_j}{c_j}\right]^2 \right) \left(\sum_{j=m+1}^n [c_j k_j]^2 \right)} \right). \tag{A.2}
 \end{aligned}$$

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